Malaysian Journal of Mathematical Sciences 20(3): 1165–1183 (2026)
https://doi.org/10.47836/mjms.20.3.14

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

Existence of a solution via $\mathcal{Z}-$ Górnicki mapping for the static deflection of a string stretched between fixed end points

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Received: 12 June 2025
Accepted: 22 January 2026

Abstract

By establishing $\mathcal{Z}-$ Górnicki mapping and proximal $\mathcal{Z}-$ Górnicki mapping via simulation function in metric spaces and partial metric spaces, we generalize Górnicki mapping and proximal Górnicki mapping in this work. To illustrate our conclusions, we offer examples to illustrate our conclusions. As an example, a solution to the second-order linear differential equation for the static deflection of a string stretched between fixed endpoints with boundary conditions is also discussed.

Keywords: fixed point, Górnicki mapping, simulation function, best proximity point, proximal Górnicki mapping

1 Introduction

Banach proposed the notion of fixed points and their uniqueness for contraction mappings on complete metric spaces. Since then, a great deal of contractions have been added by mathematicians. One such contraction is the enriched contraction, which was recently introduced by Berinde and Pacurar [3]. A mapping in this direction, the Górnicki mapping, was developed by Popescu [23] and is an extension of the enriched contraction, enriched Kannan mapping, and enriched Chatterjea mapping. Common fixed point theorems for Gornicki-type mapping are proved by Debnath [5].

Definition 1.1. [23] *Let (Γ, κ) be a complete metric spaces and $\phi : \Gamma \to \Gamma$ is said to be Górnicki mapping if*

$$\kappa(\phi\xi, \phi\omega) \leq M[\kappa(\phi\xi, \xi) + \kappa(\phi\omega, \omega) + \kappa(\xi, \omega)], \qquad \forall \xi, \omega \in \Gamma,$$

where $0 \leq M < 1$ and for all $\xi \in \Gamma$ there exists $\mu \in \Gamma$ such that

$$\kappa(\mu, \phi\mu) \leq a\, \kappa(\xi, \phi\xi),$$
$$\kappa(\xi, \mu) \leq b\, \kappa(\xi, \phi\xi),$$

for some $b \in [0, \infty)$ and $a \in [0, 1)$.

By combining several previously published findings in the literature, Khojasteh et al. [12] introduced a novel notion known as the simulation function in 2015.

Definition 1.2. [12] *Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a mapping. Then ζ is called a simulation function if it satisfies the following conditions:*

(ζ1) $\zeta(m, l) < l - m$ *for all* $t, s > 0$;

(ζ2) *if $\{m_n\}$ and $\{l_n\}$ are sequence in $(0, \infty)$ such that $\lim_{n\to\infty} m_n = \lim_{n\to\infty} l_n > 0$, then*

$$\limsup_{n\to\infty} \zeta(m_n, l_n) < 0.$$

We denote the set of all simulation functions by $\mathcal{Z}$.

Definition 1.3. [12] *Let (Γ, d) be a metric spaces, $\phi : \Gamma \to \Gamma$ a mapping and $\zeta \in \mathcal{Z}$. Then ϕ is called a $\mathcal{Z}-$ contraction with respect to ζ if the following condition satisfied:*

$$\zeta\left(\kappa(\phi\xi, \phi\omega), \kappa(\xi, \beta)\right) \geq 0 \quad \textit{for all } \xi, \omega \in \Gamma.$$

Karapinar [9] demonstrated fixed points for acceptable mapping included in simulation function in 2015. Using a simulation function, Olgun et al. [19] discovered a new class of Picard iteration. Two nonlinear operators satisfying a nonlinear contraction using a simulation function were devised by Argoubi et al. [1]. Nastasi and Vetro [16] demonstrated the existence and uniqueness of a fixed point using a lower semi-continuous function and a simulation function. Ozgur [20] demonstrated the existence of a stationary disc without making any assumptions about the continuity, completeness, or compactness of a self-mapping using a simulation function for metric spaces. Non-unique fixed point results via simulation function for the class of orbital complete b-metric spaces analyzed by Aydi et al. [2]. Fixed-point theorems demonstrated by Mlaiki et al. [15] utilizing the set of simulation functions on an S-metric space. In 2022, Rashid et al. [24] common fixed point results for q-ROF mappings are established in b-metric spaces under Suzuki-type contractive conditions. Pant et al. [22] provided a fresh approach to the Rhoades problem concerning the existence of contractive mappings that allow discontinuity at the fixed point. The simulation functions and fixed point geometric features created by Ozgur [21]. By employing a simulation function, Shagari [28] demonstrated the presence of fixed points in C^* algebra.

In 2023, Nazam et al. [17] introduced the concept of orthogonal multivalued contractions based on the recently developed notion of orthogonality in metric spaces. In 2023, nazam [17] gave generalized version of Perov's contraction principle via the Ćirić contraction principle in vector-valued b-metric spaces. In 2024, Hussain et al. [8] define a generalized Ćiric contraction with respect to $\mathfrak{T}$ and prove several coupled coincidence best proximity point theorems within the framework of complete metric spaces. In 2024, Rashid et al. [25] investigated the split fixed point problem involving multiple output sets in CAT(0) spaces. In 2024, Khan et al. [11] introduced concepts of the generalized point-to-set distance $J_\theta(\mathfrak{u}, \mathcal{H}^\star)$, the generalized Hausdorff-type distance, and the PJ_θ-property are introduced for a pair $(\mathcal{H}^\star, \mathcal{K}^\star)$ of nonempty subsets within the extended b-metric space $(\mathfrak{U}^\star, \rho_\theta)$. Hussain [7] introduced a family of mappings called dynamic interpolative contractions, and derive various fixed point results for these mappings in the setting of a complete F-metric space in 2024. Common fixed point theorems in vector-valued b-metric spaces were investigated [18], and q-ROF mappings along with Suzuki-type common fixed point results in b-metric spaces were established [24]. Recent developments in fixed point theory highlight its wide applicability in different mathematical settings. Saleem et al. [27] established approximate fixed point results for $(\alpha - \eta)$-type and $(\beta - \psi)$-type fuzzy contractive mappings in b-fuzzy metric spaces, while Wadai and Kılıçman [29] applied variational fixed point iterative schemes to analyze two-point boundary value problems.

Matthews [14] created new metric spaces in 1994 that expand upon partial metric spaces and their topological characteristics.

Definition 1.4. [14] *Let Γ be a nonempty set. A function $h : \Gamma \times \Gamma \to [0, \infty)$ is called a partial metric space if the following holds:*

(i) $h(\xi, \omega) \geq 0$ *for all* $\xi, \omega \in \Gamma$ *and* $h(\xi, \xi) = h(\xi, \omega) = h(\omega, \omega)$ *iff* $\xi = \beta$*;*

(ii) $h(\xi, \xi) \leq h(\xi, \omega)$ *for all* $\xi, \omega \in \Gamma$*;*

(iii) $h(\xi, \beta) = h(\beta, \xi)$ *for all* $\xi, \omega \in \Gamma$*;*

(iv) $h(\xi, \omega) = h(\xi, \gamma) + p(\gamma, \omega) - h(\gamma, \gamma)$ *for all* $\xi, \omega, \gamma \in \Gamma$*.*

Each partial metric p on Γ generates a ϕ_0 topology τ_h on Γ where open p-balls $\{B_h(\xi, \epsilon) : \xi \in \Gamma, \epsilon > 0\}$, where $B_h(\xi, \epsilon) = \{\omega \in \Gamma : h(\xi, \omega) < h(\xi, \xi) + \epsilon\}$ for all $\xi \in \Gamma$ and $\epsilon > 0$ similarly for closed h-ball is defined as $B_h(\xi, \epsilon) = \{\omega \in \Gamma : h(\xi, \omega) \leq h(\xi, \xi) + \epsilon\}$.

Example 1.1. [4] *Let $\Gamma = \mathbb{R}$ and $h(\alpha, \beta) = e^{\max\{\alpha, \beta\}}$ for all $\alpha, \beta \in \Gamma$. Then (Γ, h) is a partial metric space.*

Lemma 1.1. [4] *Let (Γ, h) be a partial metric space.*

(i) If $h(\xi, \omega) = 0$, then $\xi = \omega$;

(ii) If $\xi \neq \omega$, then $h(\xi, \omega) > 0$.

Lemma 1.2. [4] *Let $\xi_n \to \xi$ in a partial metric spaces (Γ, h), where $h(\xi, \xi) = 0$. Then* $\lim\limits_{n \to \infty} h(\xi_n, \omega) = h(\xi, \omega)$ *for all $\omega \in \Gamma$.*

Let Ξ and Π be a non empty subset of (Γ, κ) and $\phi : \Xi \to \Pi$ then $\xi \in \Xi$ is said to be Best Proximity point if $\phi\xi = \omega$ with $\kappa(\phi\xi, \omega) = \kappa(\Xi, \Pi)$ [26].

Using the simulation function, Kostic et al. [13] discovered the optimal proximity point in w_0 space in 2019 and Karapinar and Khojasteh [10] approximated the optimal proximity point in 2017. Proximal Górnicki mapping in metric spaces and partial metric, which is used to locate unique optimal proximity points and verify the existence of fixed points for Górnicki mappings in partial metric spaces, were introduced by Dhivya et al. in 2023 [6].

Definition 1.5. [6] *Let Ξ and Π be a nonempty subset of a complete metric space (Γ, κ). A mapping $\phi : \Xi \to \Pi$ is said to be a proximal Górnicki mapping, if for every $\xi, \omega, \mu, \nu \in \Xi$ whenever $\kappa(\phi\xi, \mu) = \kappa(\Xi, \Pi)$ and $\kappa(\phi\omega, \nu) = \kappa(\Xi, \Pi)$ implies*

$$\kappa(\mu, \nu) \leq M[\kappa(\xi, \mu) + \kappa(\xi, \omega) + \kappa(\omega, \nu)],$$

with $0 \leq M < 1$ and there exists $a, b \geq 0$ with $a < 1$ such that for given any $\xi \in \Xi$ there exists $\gamma \in \Xi$, whenever $d(\phi\gamma, \rho) = d(\Xi, \Pi)$ implies that

$$\kappa(\gamma, \rho) \leq a\kappa(\xi, \mu), \kappa(\xi, \gamma) \leq b\kappa(\xi, \mu),$$

where $\rho \in \Xi$.

Inspired by the work of [12], [23], and [6], in this paper, we introduce $\mathcal{Z}$-Górnicki mapping using simulation function in metric spaces, partial metric spaces and proves the existence and uniqueness of fixed point. In continuation of that, we are introducing proximal $\mathcal{Z}$-Górnicki mapping and proving the existence and uniqueness of the best proximity point.

2 $\mathcal{Z}$-Górnicki Mapping in Metric Spaces and Partial Metric Spaces

Here we are introducing $\mathcal{Z}$-Górnicki mapping as follows.

Definition 2.1. *Let (Γ, κ) be a metric space. $\phi : \Gamma \to \Gamma$ is said to be $\mathcal{Z}-$ Górnicki mapping with respect to $\zeta \in \mathcal{Z}$, if*

$$\zeta(\kappa(\phi\xi, \phi\omega), \kappa(\xi, \phi\xi) + \kappa(\xi, \omega) + \kappa(\omega, \phi\omega)) \geq 0 \ \forall \xi, \omega \in \Gamma, \tag{G1}$$

and also $\forall$ $\xi \in \Gamma$ there exist $\mu \in \Gamma$ such that

$$\zeta(\kappa(\mu, \phi\mu), \kappa(\xi, \phi\xi)) \geq 0, \qquad \text{and} \tag{G2}$$

$$\zeta(\kappa(\mu, \xi), \tau\, \kappa(\xi, \phi\xi)) \geq 0, \tag{G3}$$

for some $\tau \in [0, \infty)$.

Lemma 2.1. *Let (Γ, κ) be a metric space with $\phi : \Gamma \to \Gamma$ be a $\mathcal{Z} -$ Górnicki mapping with respect to $\zeta \in \mathcal{Z}$ has a fixed point. Then the fixed point must be unique.*

Proof. Suppose μ_1, μ_2 are fixed points with $\mu_1 \neq \mu_2$ by G1

$$\zeta\left(\kappa(\phi\mu_1, \phi\mu_2), \kappa(\mu_1, \phi\mu_1) + \kappa(\mu_1, \mu_2) + \kappa(\mu_2, \phi\mu_2)\right) \geq 0.$$

Then $\zeta\left(\kappa(\mu_1, \mu_2), \kappa(\mu_1, \mu_2)\right) \geq 0$, since $\kappa(\mu_1, \mu_2) \neq 0$ implies

$$0 \leq \zeta\left(\kappa(\mu_1, \mu_2), \kappa(\mu_1, \mu_2)\right) < 0,$$

from $\zeta 2$ which is a contradiction. Therefore, the fixed point must be unique. □

Theorem 2.1. *Let (Γ, κ) be a complete metric space and $\phi : \Gamma \to \Gamma$ is $\mathcal{Z} -$ Górnicki mapping with respect to $\zeta \in \mathcal{Z}$. Then ϕ has a fixed point.*

Proof. Let $\xi_0 \in \Gamma$. Then there exist $\xi_1 \in \Gamma$ such that

$$\zeta(\kappa(\xi_1, \phi\xi_1), \kappa(\xi_0, \phi\xi_0)) \geq 0, \quad \text{and}$$
$$\zeta(\kappa(\xi_1, \xi_0), \tau\,\kappa(\xi_0, \phi\xi_0)) \geq 0,$$

for some $\tau \in [0, \infty)$. In general,

$$\zeta(\kappa(\xi_{n+1}, \phi\xi_{n+1}), \kappa(\xi_n, \phi\xi_n)) \geq 0, \quad \text{and} \tag{2.1.1}$$
$$\zeta(\kappa(\xi_{n+1}, \xi_n), \tau\,\kappa(\xi_n, \phi\xi_n)) \geq 0; \tag{2.1.2}$$

case 1: (ξ_n) is sequence with for some α_k such that $\kappa(\xi_k, \phi\xi_k) = 0$, then the theorem is true.
case 2: (ξ_n) is a sequence with there is no α_k such that $\kappa(\xi_k, \phi\xi_k) = 0$, then

$$\zeta(\kappa(\xi_{n+1}, \phi\xi_{n+1}), \kappa(\xi_n, \phi\xi_n)) \geq 0,$$

implies $\kappa(\xi_{n+1}, \phi\xi_{n+1}) < \kappa(\xi_n, \phi\xi_n)$. From $\zeta 2$, which shows that $\{\kappa(\xi_n, \phi\xi_n)\}$ is decreasing sequence. So there exist $c \in \mathbb{R}$ such that $\lim_{n\to\infty} \kappa(\xi_n, \phi\xi_n) = c$. Suppose $c > 0$ and also

$$0 \leq \limsup \zeta(\kappa(\phi\alpha_{n+1}, \alpha_{n+1}), \kappa(\xi_n, \phi\xi_n)) < 0.$$

This contradiction shows $\lim_{n\to\infty} \kappa(\phi\xi_n, \xi_n) = 0$.
Now, suppose for some $n \in \mathbb{N}$, $\kappa(\xi_n, \xi_{n+1}) = 0$. From 2.1.1 and $\zeta 2$

$$0 \leq \zeta\left(\kappa(\xi_n, \phi\xi_n), \kappa(\xi_n, \phi\xi_n)\right) < 0.$$

This contradiction implies $d(\xi_n, \xi_{n+1}) \neq 0,\ \forall\ n \in \mathbb{N}$. From 2.1.2 and $\zeta 2$, we have $\kappa(\xi_{n+1}, \xi_n) < \tau\,\kappa(\xi_n, \phi\xi_n)$. Therefore $\lim_{n\to\infty} \kappa(\xi_n, \xi_{n+1}) = 0$.
claim: (ξ_n) is bounded.
Suppose (ξ_n) is not bounded. Then there exists subsequence (ξ_{n_k}) such that $\kappa(\xi_{n_k}, \alpha_{n_{k+1}}) > 1$ and $\kappa(\xi_m, \xi_{n_k}) \leq 1$ for $n_k \leq m \leq n_{k+1} - 1$ implies

$$\begin{aligned} 1 < \kappa(\xi_{n_{k+1}}, \xi_{n_k}) &\leq \kappa(\xi_{n_{k+1}}, \xi_{n_{k+1}-1}) + \kappa(\xi_{n_{k+1}-1}, \xi_{n_k}), \\ &\leq \kappa(\xi_{n_{k+1}}, \xi_{n_{k+1}-1}) + 1. \end{aligned}$$

Taking limit, we have $\lim_{k\to\infty} \kappa(\xi_{n_k}, \xi_{n_{k+1}}) = 1$.
Also

$$\kappa(\phi\xi_{n_{k+1}}, \phi\xi_{n_k}) \leq \kappa(\phi\xi_{n_{k+1}}, \xi_{n_{k+1}}) + \kappa(\xi_{n_k}, \xi_{n_{k+1}}) + \kappa(\xi_{n_k}, \phi\xi_{n_k}).$$

Applying $\limsup$ on both sides implies

$$\limsup \kappa(\phi\xi_{n_{k+1}}, \phi\xi_{n_k}) \leq 1.$$

Also, we have

$$\kappa(\xi_{n_{k+1}}, \xi_{n_k}) \leq d(\xi_{n_{k+1}}, \phi\xi_{n_{k+1}}) + \kappa(\phi\xi_{n_{k+1}}, \phi\xi_{n_k}) + \kappa(\phi\xi_{n_k}, \xi_{n_k}).$$

Taking $\liminf$ on both sides implies

$$1 \leq \liminf \kappa(\phi\alpha_{n_{k+1}}, \phi\xi_{n_k}).$$

Therefore, $\lim_{k\to\infty} \kappa(\phi\xi_{n_{k+1}}, \phi\xi_{n_k}) = 1$. Also, we can say

$$\lim_{k\to\infty} \kappa(\xi_{n_k}, \phi\xi_{n_k}) + \kappa(\xi_{n_k}, \xi_{n_{k+1}}) + \kappa(\xi_{n_{k+1}}, \phi\xi_{n_{k+1}}) = 1.$$

From G1, we have

$$0 \leq \limsup \zeta \left(\kappa(\phi\xi_{n_{k+1}}, \phi\xi_{n_k}), \kappa(\xi_{n_k}, \phi\xi_{n_k}) + \xi(\xi_{n_k}, \xi_{n_{k+1}}) + \kappa(\xi_{n_{k+1}}, \phi\xi_{n_{k+1}})\right) < 0.$$

This contradiction implies (ξ_n) is bounded. Now we prove that (ξ_n) is cauchy sequence. Consider $c_n = \sup\{\kappa(\xi_i, \xi_j) \mid i, j \geq n\}$. Since (ξ_n) is bounded implies c_n exist and it is monotonically decreasing. Suppose $\lim\limits_{n\to\infty} c_n = c > 0$. Then there exist subsequence $(n_k), (m_k)(l_k) \in \mathbb{N}$ such that

$$c_{l_k} - \frac{1}{k} < \kappa(\xi_{m_k}, \xi_{n_k}) \leq c_{l_k}.$$

By taking $\lim$, we have

$$\lim_{k\to\infty} \kappa(\xi_{m_k}, \xi_{n_k}) = c.$$

Therefore

$$\lim_{k\to\infty} \kappa(\xi_{m_k}, \xi_{n_k}) + \kappa(\xi_{m_k}, \phi\xi_{m_k}) + \kappa(\xi_{n_k}, \phi\xi_{n_k}) = c.$$

Now, consider

$$\kappa(\xi_{m_k}, \xi_{n_k}) \leq \kappa(\xi_{m_k}, \phi\xi_{m_k}) + \kappa(\phi\xi_{m_k}, \phi\xi_{n_k}) + \kappa(\xi_{n_k}, \phi\xi_{n_k}).$$

Taking $\liminf$ on both sides implies

$$c \leq \liminf \kappa(\phi\xi_{m_k}, \phi\xi_{n_k}).$$

Also, we have

$$\kappa(\phi\xi_{m_k}, \phi\xi_{n_k}) \leq \kappa(\phi\xi_{m_k}, \xi_{m_k}) + \kappa(\xi_{m_k}, \xi_{n_k}) + \kappa(\phi\xi_{n_k}, \xi_{n_k}).$$

Taking $\limsup$ on both side implies

$$\limsup \kappa(\phi\xi_{m_k}, \phi\xi_{n_k}) \leq c,$$

so we can say $\lim\limits_{k\to\infty} \kappa(\phi\xi_{m_k}, \phi\xi_{n_k}) = c$.
From G1 and $\zeta 3$

$$0 \leq \limsup \zeta \left(\kappa(\phi\xi_{m_k}, \phi\xi_{n_k}), \kappa(\xi_{m_k}, \xi_{n_k}) + \kappa(\xi_{m_k}, \phi\xi_{m_k}) + \kappa(\xi_{n_k}, \phi\xi_{m_k})\right) < 0,$$

which is a contradiction. Therefore, (ξ_n) is cauchy. Since Γ is complete, there exist μ in Γ such that $\lim\limits_{n\to\infty} (\xi_n) = \mu$.

By the triangular inequality, $\kappa(\phi\xi_n, \mu) \leq \kappa(\xi_n, \mu) + \kappa(\xi_n, \phi\xi_n)$, we can say $\lim\limits_{n\to\infty} \kappa(\phi\xi_n, \mu) = 0$.

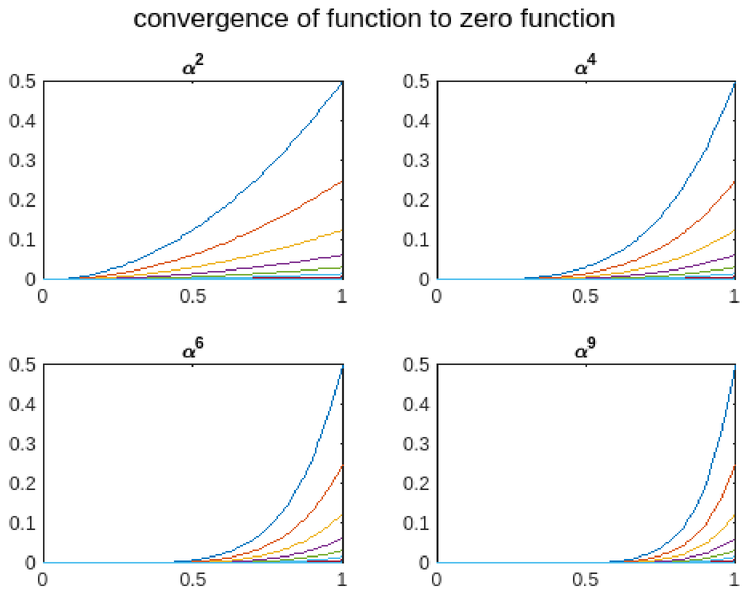
Suppose $\phi\mu \neq \mu$. From G1

$$0 \leq \limsup \zeta \left(\kappa(\phi\xi_n, \phi\mu), \kappa(\xi_n, \phi\xi_n) + \kappa(\phi\mu, \mu) + \kappa(\xi_n, \mu)\right) < 0,$$

which implies $\phi\mu = \mu$. □

Example 2.1. *Let $\mathcal{F} = \{f : [0,1] \to [0,1] \mid f$ is continuous$\}$ with $\|f\| = \sup|f(\xi)|$ and $\phi : \mathcal{F} \to \mathcal{F}$ such that $\phi f = \frac{f}{4}\, \forall f \in \mathcal{F}$. Then ϕ is $\mathcal{Z} -$ Górnicki mapping with respect to some $\zeta \in \mathcal{Z}$.*

Proof. In this case ϕ is $\mathcal{Z} -$ *Górnicki mapping* with respect to $\zeta = \frac{9}{10}s - t$ with $\tau = 1$.
Claim: $\zeta\left(\kappa(\phi f, \phi g), \kappa(f, \phi f) + \kappa(g, \phi g) + \kappa(f, g)\right) \geq 0$, consider $l = \sup f(\xi)$, $m = \sup g(\xi)$, $r = \inf f(\xi)$ and $s = \inf g(\xi)$.
Suppose $l \geq m$. Then



$$\frac{9}{10}s - t = \frac{9}{10}\left(\left\|f - \frac{f}{4}\right\| + \left\|g - \frac{g}{4}\right\| + \|f - g\|\right) - \left\|\frac{f}{4} - \frac{g}{4}\right\|,$$
$$\geq \frac{9}{10}\left(\frac{3l}{4} + \frac{3m}{4}\right) - \frac{l}{4} + \frac{s}{4} \geq 0.$$

Suppose $m \geq l$. Then

$$\frac{9}{10}s - t \geq \frac{9}{10}\left(\frac{3l}{4} + \frac{3m}{4}\right) - \frac{m}{4} + \frac{r}{4} \geq 0.$$

Claim: For every $f \in \mathcal{F}$ there exists $g \in \mathcal{F}$ such that $\zeta\left(\|f-g\|, \|f-\phi f\|\right) \geq 0$ and $\zeta\left(\|\phi g - g\|, \|f - \phi f\|\right) \geq 0$.
Consider $g = \frac{f}{2}$. Then

$$\frac{9}{10}s - t = \frac{9}{10}\left\|f - \frac{f}{4}\right\| - \left\|f - \frac{f}{2}\right\| = \frac{7}{40}\sup|f(\xi)| \geq 0,$$

and also

$$\frac{9}{10}s - t = \frac{9}{10}\left\|f - \frac{f}{4}\right\| - \left\|\frac{f}{8} - \frac{f}{2}\right\| = \frac{12}{40}\sup|f(\xi)| \geq 0.$$

□

Remark 2.1. *Suppose (Γ, κ) be a metric space and $\phi : \Gamma \to \Gamma$ to say ϕ is $\mathcal{Z}-contraction$ it is necessary to be continuous but for ϕ to be $\mathcal{Z}$-Górnicki mapping the function doesn't need to be continuous.*

Corollary 2.1. *Let (Γ, κ) is a metric space and $\phi : \Gamma \to \Gamma$, there exists $\psi : [0, \infty) \to [0, 1)$ with $\limsup_{t\to r} \psi(t) < 1\ \forall r > 0$ such that*

$$\kappa(\phi\xi, \phi\omega) \leq \psi\left(\kappa(\phi\xi, \phi\beta)\right)\left(\kappa(\phi\xi, \xi) + \kappa(\phi\omega, \omega) + \kappa(\xi, \omega)\right), \qquad \forall\, \xi, \omega \in \Gamma,$$

and for every $\alpha \in \Gamma$ there exists $\mu \in \Gamma$ such that

$$\kappa(\mu, \xi) \leq \psi\left(\kappa(\mu, \xi)\right)\left(\kappa(\phi\xi, \xi)\right),$$
$$\kappa(\phi\mu, \mu) \leq \psi\left(\kappa(\phi\mu, \mu)\right)\left(\kappa(\phi\xi, \xi)\right).$$

Then ϕ is $\mathcal{Z}$-Gornicki mapping with some $\zeta \in \mathcal{Z}$.

Proof. Consider $\zeta(t,s) = \psi(t)s - t$ then ϕ is $\mathcal{Z}-$ Górnicki mapping with respect to ζ, where $\tau = 1$. □

Corollary 2.2. *Suppose $\phi : \Gamma \to \Gamma$ is Górnicki mapping. Then ϕ is $\mathcal{Z}-$ Górnicki mapping with some $\zeta \in \mathcal{Z}$.*

Proof. Use $\psi : [0,\infty) \to [0,1)$ such that $\psi(t) = max\{M,a\} = c\ \forall\ t \in [0,\infty)$ suppose $\max\{M,a\} = 0$ take $c \in (0,1)$ in corollary 2.1 with $\tau = \frac{b}{c}$. □

Theorem 2.2. *Let (Γ, d) be a complete metric spaces, $\phi : \Gamma \to \Gamma$ with $\psi : \Gamma \to [0,\infty)$ such that*

$$\zeta\left(\kappa(\phi\xi,\phi\omega) + \psi(\phi\xi) + \psi(\phi\omega), \kappa(\xi,\phi\xi) + \kappa(\omega,\phi\omega) + \kappa(\xi,\omega) + \psi(\xi) + \psi(\phi\omega)\right) \geq 0$$

for all $\xi, \beta \in \Gamma$ and also for every $\xi \in \Gamma$ there exist $\mu \in \Gamma$ such that

$$\zeta\left(\kappa(\xi,\mu) + \psi(\xi) + \psi(\mu), \tau\,\kappa(\phi\xi,\xi) + \psi(\phi\xi) + \psi(\xi)\right) \geq 0, \text{ and}$$
$$\zeta\left(\kappa(\phi\mu,\mu) + \psi(\phi\mu) + \psi(\mu), \kappa(\phi\xi,\xi) + \psi(\phi\xi) + \psi(\xi)\right) \geq 0,$$

where $\tau \in [0,\infty)$. Then ϕ has unique fixed point.

Proof. From the assumption there exist sequence (α_n) such that

$$\zeta\left(\kappa(\xi_n,\xi_{n+1}) + \psi(\xi_n) + \psi(\xi_{n+1}), \tau\,\kappa(\phi\xi_n,\xi_n) + \psi(\phi\xi_n) + \psi(\xi_n)\right) \geq 0, \text{ and}$$
$$\zeta\left(\kappa(\phi\xi_{n+1},\xi_{n+1}) + \psi(\phi\xi_{n+1}) + \psi(\xi_{n+1}), \kappa(\phi\xi_n,\xi_n) + \psi(\phi\xi_n) + \psi(\xi_n)\right) \geq 0.$$

Similar to theorem 2.1 we can say (α_n) is Cauchy sequence with $\lim_{n\to\infty} \kappa(\xi_n,\xi_{n+1}) = 0$, $\lim_{n\to\infty} \kappa(\xi_n,\phi\xi_n) = 0$, also $\lim_{n\to\infty} \psi(\phi\alpha_n) = 0$ and $\lim_{n\to\infty} \psi(\xi_n) = 0$. Since Γ is complete there exists $\mu \in \Gamma$ such that (ξ_n) converges to μ and from the hypothesis of the theorem

$$\zeta\left(\kappa(\phi\xi_n,\phi\mu) + \psi(\phi\xi_n) + \psi(\phi\mu), \kappa(\xi_n,\phi\xi_n) + \kappa(\mu,\phi\mu)\ \kappa(\xi_n,\mu) + \psi(\xi_n) + \psi(\phi\mu)\right) \geq 0.$$

Suppose $\kappa(\mu,\phi\mu) \neq 0$. From $\zeta 3$

$$0 \leq \limsup \zeta\left(\kappa(\phi\xi_n,\phi\mu) + \psi(\phi\xi_n) + \psi(\phi\mu)\ , \kappa(\xi_n,\phi\xi_n) + \kappa(\mu,\phi\mu) + \kappa(\xi_n,\mu) + \psi(\xi_n) + \psi(\phi\mu)\right) < 0.$$

Therefore $\phi\mu = \mu$ and uniqueness is easy to prove. □

Remark 2.2. *Theorem 2.1 is special case of Theorem 2.2*

Definition 2.2. *Let (Γ, h) be a partial metric spaces and $\phi : \Gamma \to \Gamma$ is said to be partial $\mathcal{Z}-$ Górnicki mapping with respect to $\zeta \in \mathcal{Z}$ if*

$$\zeta(h(\phi\xi,\phi\beta), h(\xi,\phi\xi) + h(\xi,\beta) + h(\beta,\phi\beta)) \geq 0, \quad \forall \xi,\beta \in \Gamma, \tag{PG1}$$

and also $\forall\ \xi \in \Gamma$ there exist $\mu \in \Gamma$ such that

$$\zeta\left(h(\mu,\phi\mu), h(\xi,\phi\xi)\right) \geq 0, \quad \text{and} \tag{PG2}$$
$$\zeta\left(h(\mu,\xi), \tau\, h(\xi,\phi\xi)\right) \geq 0, \tag{PG3}$$

for some $\tau \in [0,\infty)$.

Theorem 2.3. *Let (Γ, h) be a complete partial metric spaces and $\phi : \Gamma \to \Gamma$ is partial $\mathcal{Z}-$ Górnicki mapping with respect to $\zeta \in \Gamma$. Then ϕ has atleast one fixed point.*

Proof. Consider $\xi_0 \in \Gamma$, there exist $\xi_1 \in \Gamma$ such that

$$\zeta\left(h(\xi_1, \xi_0), \tau\, h(\xi_0, \phi\xi_0)\right) \geq 0, \quad \text{and}$$
$$\zeta\left(h(\xi_1, \phi\xi_1), h(\xi_0, \phi\xi_0)\right) \geq 0.$$

In general

$$\zeta\left(h(\xi_{n+1}, \xi_n), \tau\, h(\xi_n, \phi\xi_n)\right) \geq 0, \quad \text{and} \tag{2.3.1}$$
$$\zeta\left(h(\xi_{n+1}, \phi\xi_{n+1}), h(\xi_n, \phi\xi_n)\right) \geq 0. \tag{2.3.2}$$

Case 1: (ξ_n) is a sequence with some ξ_k such that $h(\xi_k, \phi\xi_k) = 0$. Then $\xi_k = \phi\xi_k$.
Case 2: (α_n) is a sequence with no α_k such that $h(\xi_k, \phi\xi_k) = 0$. Then from 2.3.2,

$$0 \leq \zeta\left(h(\xi_{n+1}, \phi\xi_{n+1}), h(\alpha_n, \phi\xi_n)\right) < h(\xi_n, \phi\xi_n) - h(\xi_{n+1}, \phi\xi_{n+1}),$$

implies $h(\xi_{n+1}, \phi\xi_{n+1}) < h(\xi_n, \phi\xi_n)$. This shows that $\{h(\xi_n, \phi\xi_n)\}$ is a decreasing sequence. Suppose $\lim_{n\to\infty} h(\xi_n, \phi\xi_n) = c > 0$. Then from $\zeta 3$,

$$0 \leq \limsup \zeta\left(h(\xi_{n+1}, \phi\xi_{n+1}), h(\xi_n, \phi\xi_n)\right) < 0.$$

This contradiction implies $\lim_{n\to\infty} h(\xi_n, \phi\xi_n) = 0$. Suppose $h(\xi_n, \xi_{n+1}) = 0$ for some $n \in \mathbb{N}$. From 2.3.2 and $\zeta 2$

$$0 \leq \zeta\left(h(\xi_n, \phi\xi_n), h(\xi_n, \phi\xi_n)\right) < 0.$$

So we can say $h(\alpha_n, \alpha_{n+1}) \neq 0$ for any $n \in \mathbb{N}$ from 2.3.1 $h(\xi_{n+1}, \xi_n) < \tau\, h(\xi_n, \phi\xi_n)$ implies $\lim_{n\to\infty} h(\xi_n, \xi_{n+1}) = 0$.
Claim: (ξ_n) is bounded with respect to metric $\kappa_h(\xi, \omega) = 2h(\xi, \omega) - h(\xi, \xi) - h(\omega, \omega)$
It is clear that $\lim_{n\to\infty} \kappa_h(\xi_n, \xi_{n+1}) = 0$ and $\lim_{n\to\infty} \kappa_h(\phi\xi_n, \xi_n) = 0$. Suppose (ξ_n) is not bounded. There exists (ξ_{n_k}) such that $\kappa_h(\xi_{n_k}, \xi_{n_{k+1}}) > 1$ and $\kappa_h(\xi_m, \xi_{n_k}) \leq 1$ for all $n_k \leq m \leq n_{k+1} - 1$.

Consider

$$1 < \kappa_h(\xi_{n_{k+1}}, \xi_{n_k}) \leq \kappa_h(\xi_{n_{k+1}}, \xi_{n_{k+1}-1}) + \kappa_h(\xi_{n_{k+1}-1}, \xi_{n_k}), \text{ and}$$
$$1 < \kappa_h(\xi_{n_{k+1}}, \xi_{n_k}) \leq \kappa_h(\xi_{n_{k+1}}, \xi_{n_{k+1}-1}) + 1,$$

applying limit on both the side implies $\lim_{n\to\infty} \kappa_h(\xi_{n_{k+1}}, \xi_{n_k}) = 1$, and also

$$\kappa_h(\phi\xi_{n_{k+1}}, \phi\xi_{n_k}) \leq \kappa_h(\phi\xi_{n_{k+1}}, \xi_{n_{k+1}}) + \kappa_h(\xi_{n_{k+1}}, \xi_{n_k}) + \kappa_h(\xi_{n_k}, \phi\xi_{n_k}).$$

Applying $\limsup$ on both sides, we get

$$\limsup \kappa_h(\phi\xi_{n_{k+1}}, \phi\xi_{n_k}) \leq 1,$$

and

$$\kappa_h(\xi_{n_{k+1}}, \xi_{n_k}) \leq \kappa_h(\xi_{n_{k+1}}, \phi\xi_{n_{k+1}}) + \kappa_h(\phi\xi_{n_{k+1}}, \phi\xi_{n_k}) + \kappa_h(\phi\xi_{n_k}, \xi_{n_k}).$$

Taking $\liminf$ on both sides, we get

$$1 \leq \liminf \kappa_h(\phi\xi_{n_{k+1}}, \phi\xi_{n_k}),$$

which implies

$$\lim_{n\to\infty} \kappa_h(\phi\xi_{n_{k+1}}, \phi\xi_{n_k}) = 1.$$

Now, we can say

$$\lim_{n\to\infty} \kappa_h(\xi_{n_k}, \phi\xi_{n_k}) + \kappa_h(\xi_{n_{k+1}}, \phi\xi_{n_{k+1}}) + \kappa_h(\phi\xi_{n_k}, \phi\xi_{n_{k+1}}) = 1.$$

So we have

$$\lim_{n\to\infty} h(\phi\xi_{n_k}, \phi\xi_{n_{k+1}}) = \frac{1}{2}, \quad \text{and}$$
$$\lim_{n\to\infty} h(\xi_{n_k}, \phi\xi_{n_k}) + h(\xi_{n_k}, \xi_{n_{k+1}}) + h(\xi_{n_{k+1}}, \phi\xi_{n_{k+1}}) = \frac{1}{2}.$$

From $\zeta 3$

$$0 \leq \limsup \zeta\left(h(\phi\xi_{n_k}, \phi\xi_{n_{k+1}}), h(\xi_{n_k}, \phi\xi_{n_k}) + h(\xi_{n_k}, \xi_{n_{k+1}}) + h(\xi_{n_{k+1}}, \phi\xi_{n_{k+1}})\right) < 0.$$

This contradiction implies (ξ_n) is bounded.
Now, we prove that (ξ_n) is cauchy. Consider $c_n = \sup\{\kappa_h(\xi_i, \xi_j) | i, j \geq n\}$, clearly c_n is monotonically decreasing sequence. Suppose $\lim_{n\to\infty} c_n = c > 0$. There exist subsequence n_k, m_k, l_k in $\mathbb{N}$ such that

$$c_{l_k} - \frac{1}{k} < \kappa_h(\xi_{m_k}, \xi_{n_k}) \leq c_{l_k}.$$

Then $\kappa_h(\xi_{m_k}, \xi_{n_k}) = c$. Also

$$\kappa_h(\phi\xi_{m_k}, \phi\xi_{n_k}) \leq \kappa_h(\phi\xi_{m_k}, \xi_{m_k}) + \kappa_h(\xi_{m_k}, \xi_{n_k}) + \kappa_h(\xi_{n_k}, \phi\alpha_{n_k}).$$

Applying $\limsup$ on both side

$$\limsup \kappa_h(\phi\xi_{m_k}, \phi\xi_{n_k}) \leq c,$$

and

$$d_h(\xi_{m_k}, \xi_{n_k}) \leq \kappa_h(\xi_{m_k}, \phi\xi_{m_k}) + \kappa_h(\phi\xi_{m_k}, \phi\xi_{n_k}) + \kappa_h(\phi\alpha_{n_k}, \xi_{n_k}).$$

Applying $\liminf$ on both side

$$c \leq \liminf \kappa_h(\phi\xi_{m_k}, \phi\xi_{n_k}).$$

Therefore $\lim_{k\to\infty} \kappa_h(\phi\xi_{m_k}, \phi\xi_{n_k}) = c$. Also, we have

$$\lim_{k\to\infty} \kappa_h(\xi_{m_k}, \alpha_{n_k}) + \kappa_h(\xi_{m_k}, \phi\xi_{m_k}) + \kappa_h(\alpha_{n_k}, \phi\xi_{n_k}) = c.$$

We can also say

$$\lim_{k\to\infty} h(\phi\xi_{m_k}, \phi\xi_{n_k}) = \frac{c}{2},$$
$$\lim_{k\to\infty} h(\xi_{m_k}, \xi_{n_k}) + h(\xi_{m_k}, \phi\xi_{m_k}) + h(\xi_{n_k}, \phi\xi_{n_k}) = \frac{c}{2}.$$

From $\zeta 3$

$$0 \leq \limsup \zeta\left(h(\phi\xi_{m_k}, \phi\xi_{n_k}), h(\xi_{m_k}, \xi_{n_k}) + h(\xi_{m_k}, \phi\xi_{m_k}) + h(\alpha_{n_k}, \phi\xi_{n_k})\right) < 0.$$

Therefore (ξ_n) is cauchy with $\lim_{n,m\to\infty} h(\xi_n, \xi_m) = 0$. Since (Γ, h) is complete partial metric space, there exist $\mu \in \Gamma$ such that

$$\lim_{n\to\infty} h(\xi_n, \mu) = \lim_{n,m\to\infty} h(\xi_n, \xi_m) = h(\mu, \mu) = 0.$$

Hence

$$h(\mu, \mu) \leq h(\phi\xi_n, \mu) \leq h(\xi_n, \phi\xi_n) + h(\xi_n, \mu).$$

Applying limit on both side implies $\lim_{n\to\infty} h(\phi\xi_n, \mu) = 0$. From lemma 1.2
$\lim_{n\to\infty} h(\phi\xi_n, \phi\mu) = h(\mu, \phi\mu)$, and also $\lim_{n\to\infty} h(\xi_n, \phi\xi_n) + h(\mu, \phi\mu) + h(\xi_n, \mu) = h(\phi\mu, \mu)$.
Suppose $h(\phi\mu, \mu) \neq 0$. Then

$$0 \leq \limsup \zeta\left(h(\phi\xi_n, \phi\mu), h(\xi_n, \phi\xi_n) + h(\mu, \phi\mu) + h(\xi_n, \mu)\right) < 0.$$

This contradiction implies $h(\mu, \phi\mu) = 0$. Therefore $\phi\mu = \mu$. □

Remark 2.3. *If self distance of every fixed point of ϕ is 0, then the uniqueness of the above theorem holds by $\zeta 2$ and PG1.*

Example 2.2. *Let (Γ, h) be a partial metric spaces with $\Gamma = [0,\infty)$ and $h(\xi,\beta) = \max\{\xi,\omega\}$. Consider $\phi : \Gamma \to \Gamma$ such that $\phi\xi = \frac{10}{9}\xi$, then ϕ is partial $\mathcal{Z}-$ Górnicki mapping with $\zeta(t,s) = \frac{19}{20}s - t$.*

Proof. Suppose $\xi \geq \omega$,

$$\frac{19}{20}\left(\max\left\{\xi, \frac{10}{9}\xi\right\} + \max\left\{\omega, \frac{10}{9}\omega\right\} + \max\{\xi,\omega\}\right) - \max\left\{\frac{10}{9}\xi, \frac{10}{9}\omega\right\} = \begin{cases} \frac{161}{180}\xi + \frac{19}{18}\omega \geq 0 & \xi \geq \omega \\ \frac{161}{180}\omega + \frac{19}{18}\xi \geq 0 & \xi < \omega \end{cases},$$

take $\mu = \frac{\xi}{2}$. If $\xi \in [0,\infty)$,

$$\frac{19}{20}\max\left\{\xi, \frac{10}{9}\xi\right\} - \max\left\{\xi, \frac{\xi}{2}\right\} = \frac{19}{18}\xi - \xi \geq 0, \text{ and}$$
$$\frac{19}{20}\max\left\{\xi, \frac{10}{9}\xi\right\} - \max\left\{\frac{\xi}{2}, \frac{10}{18}\xi\right\} = \frac{19}{18}\xi - \frac{10}{18}\xi \geq 0.$$

Therefore, ϕ is $\mathcal{Z}-$ Górnicki mapping with respect to $\zeta(t,s) = \frac{19}{20}s - t$, where $\tau = 1$. □

Theorem 2.4. *Let (Γ, h) be a partial complete metric space, $\phi : \Gamma \to \Gamma$ with $\psi : \Gamma \to [0,\infty)$, where ψ is continuous and*

$$\zeta\left(h(\phi\xi,\phi\omega) + \psi(\phi\xi) + \psi(\phi\omega), h(\xi,\phi\xi) + h(\omega,\phi\omega) + h(\xi,\omega) + \psi(\xi) + \psi(\phi\omega)\right) \geq 0$$

for all $\xi,\beta \in \Gamma$ and also for every $\xi \in \Gamma$ there exist $\mu \in \Gamma$ such that

$$\zeta\left(h(\xi,\mu) + \psi(\xi) + \psi(\mu), \tau h(\phi\xi,\xi) + \psi(\phi\xi) + \psi(\xi)\right) \geq 0, \quad \textit{and}$$
$$\zeta\left(h(\phi\mu,\mu) + \psi(\phi\mu) + \psi(\mu), h(\phi\xi,\xi) + \psi(\phi\xi) + \psi(\xi)\right) \geq 0.$$

Then ϕ has a unique fixed point.

Proof. Use a procedure similar to Theorem 2.2 and Theorem 2.3. □

Remark 2.4. *Theorem 2.3 is a special case of Theorem 2.4.*

Corollary 2.3. *Let (Γ, h) be a complete partial metric space $\phi : \Gamma \to \Gamma$ and there exists $\psi : [0,\infty) \to [0,1)$ with $\limsup_{t\to r} = \psi(t) < 1, \forall r > 0$ such that*

$$h(\phi\xi,\phi\omega) \leq \psi\left(h(\phi\xi,\phi\omega)\right) h(\phi\xi,\xi) + h(\phi\omega,\omega) + h(\xi,\omega),$$

and there exists $\mu \in \Gamma$ such that

$$h(\mu,\xi) \leq \psi\left(h(\mu,\xi)\right) h(\phi\xi,\xi), \quad \textit{and}$$
$$h(\phi\mu,\mu) \leq \psi\left(h(\phi\mu,\mu)\right) h(\phi\xi,\xi).$$

Then ϕ is $\mathcal{Z}$-Górnicki with some $\zeta \in \mathcal{Z}$ where $\tau \in [0,\infty)$.

Proof. similar to corollary 2.1. □

Corollary 2.4. *Let (Γ, h) be a partial metric space $\phi : \Gamma \to \Gamma$ is partial Górnicki mapping with $b < 1$. Then ϕ is partial $\mathcal{Z}-$ Górnicki mapping with some $\zeta \in \mathcal{Z}$.*

Proof. Use $\psi : [0, \infty) \to [0, 1)$ such that $\psi(t) = \max(M, a) = c, \ \forall\ t \in [0, \infty)$. Suppose $\max\{M, a\} = 0$, and take $c \in (0, 1)$ in corollary 2.3 with $\tau = \dfrac{b}{c}$. □

3 Best Proximity Point in Proximal $\mathcal{Z}$-Górnicki Mapping

Definition 3.1. *Let Ξ and Π be nonempty subsets of complete metric space (Γ, κ). A mapping $\phi : \Xi \to \Pi$ is said to be proximal $\mathcal{Z}-$ Górnicki mapping with respect to ζ in $\mathcal{Z}$ if for every $\xi, \omega, \mu, \nu \in \Xi$ whenever*

$$\kappa(\phi\xi, \mu) = \kappa(\Xi, \Pi), \quad \textit{and}$$
$$\kappa(\phi\omega, \nu) = \kappa(\Xi, \Pi),$$

imply

$$\zeta\left(\kappa(\mu, \nu), \kappa(\xi, \mu) + \kappa(\xi, \omega) + \kappa(\omega, \nu)\right) \geq 0, \tag{BZ1}$$

there exist $\tau \in [0, \infty)$ such that for given any $\xi \in \Xi$, we have $\gamma \in \Xi$ whenever $\kappa(\phi\gamma, \rho) = \kappa(\Xi, \Pi)$ implies that

$$\zeta\left(\kappa(\gamma, \rho), \kappa(\xi, \mu)\right) \geq 0, \quad \textit{and} \tag{BZ2}$$
$$\zeta\left(\kappa(\xi, \gamma), \tau\, \kappa(\xi, \mu)\right) \geq 0, \tag{BZ3}$$

where $\rho \in \Xi$.

Lemma 3.1. *Let Ξ and Π be a nonempty subsets of a complete metric space (Γ, κ). If a mapping $\phi : \Xi \to \Pi$ which is proximal $\mathcal{Z}-$ Górnicki mapping with respect to ζ has the best proximity point, then it must be unique.*

Proof. Let μ_1, μ_2 be best proximity points of ϕ with $\mu_1 \neq \mu_2$. Then $\kappa(\phi\mu_1, \mu_1) = \kappa(\Xi, \Pi)$ and $\kappa(\phi\mu_2, \mu_2) = \kappa(\Xi, \Pi)$. From BZ1 and $\zeta 2$, we have

$$0 \leq \zeta\left(\kappa(\mu_1, \mu_2), \kappa(\mu_1, \mu_2)\right) < 0.$$

Therefore, the best proximity point must be unique. □

Theorem 3.1. *Let Ξ and Π be nonempty closed subsets of complete metric space (Γ, κ). If a mapping $\phi : \Xi \to \Pi$ which is proximal $\mathcal{Z}-$ Górnicki mapping with respect to ζ, then ϕ has a best proximity point.*

Proof. Let $\xi_0 \in \Xi$. There exists $\mu_0 \in \Xi$ such that $\kappa(\phi\xi_0, \mu_0) = \kappa(\Xi, \Pi)$. Then with respect to $\xi_0, \mu_0 \in \Xi$ there exist $\xi_1, \mu_1 \in \Xi$ such that

$$\zeta\left(\kappa(\xi_1, \mu_1), \kappa(\xi_0, \mu_0)\right) \geq 0, \quad \text{and}$$
$$\zeta\left(\kappa(\xi_0, \xi_1), \tau\kappa(\xi_0, \mu_0)\right) \geq 0,$$

where $\kappa(\phi\xi_1, \mu_1) = \kappa(\Xi, \Pi)$. In general

$$\zeta\left(\kappa(\xi_{n+1}, \mu_{n+1}), \kappa(\xi_n, \mu_n)\right) \geq 0, \quad \text{and} \tag{3.5.1}$$
$$\zeta\left(\kappa(\xi_n, \xi_{n+1}), \tau\kappa(\xi_n, \mu_n)\right) \geq 0, \tag{3.5.2}$$

where $\kappa(\phi\xi_n, \mu_n) = \kappa(\Xi, \Pi)$. Suppose $\kappa(\xi_n, \mu_n) = 0$. Then $\xi_n \in \Xi$ is best proximity point since we have $\kappa(\phi\xi_n, \mu_n) = \kappa(\Xi, \Pi)$. So let us consider $\kappa(\xi_n, \mu_n) \neq 0$, $\forall n \in \mathbb{N}$, from 3.5.1, $\zeta 2$ and $\zeta 3$

$$\lim_{n\to\infty} \kappa(\xi_n, \mu_n) = 0.$$

If $\kappa(\xi_n, \xi_{n+1}) \neq 0$, then $\kappa(\xi_n, \xi_{n+1}) < \tau\, \kappa(\xi_n, \mu_n)$ implies

$$\lim_{n\to\infty} \kappa(\xi_n, \xi_{n+1}) = 0.$$

Similar to theorem 2.1 we can say (ξ_n) is cauchy sequence. Since Ξ is closed subset of complete metric space, there exist $\gamma \in A$ such that

$$\lim_{n\to\infty} \alpha_n = \gamma.$$

Also, from triangular inequality $\kappa(\mu_n, \gamma) \leq \kappa(\xi_n, \mu_n) + \kappa(\xi_n, \gamma)$, we can say

$$\lim_{n\to\infty} \mu_n = \gamma.$$

By definition there exists $\nu \in \Xi$ such that $\kappa(\phi\gamma, \nu) = \kappa(\Xi, \Pi)$. Since we have $\kappa(\phi\xi_n, \mu_n) = \kappa(\Xi, \Pi)$, applying BZ1

$$0 \leq \zeta\left(\kappa(\mu_n, \nu), \kappa(\xi_n, \mu_n) + \kappa(\gamma, \nu) + \kappa(\xi_n, \gamma)\right).$$

Suppose $\gamma \neq \nu$. Then

$$0 \leq \limsup \zeta\left(\kappa(\mu_n, \nu), \kappa(\xi_n, \mu_n) + \kappa(\gamma, \nu) + \kappa(\xi_n, \gamma)\right) < 0,$$

which is a contradiction. Therefore $\kappa(\phi\gamma, \gamma) = \kappa(\Xi, \Pi)$. □

Example 3.1. *consider $\phi : \Xi \to \Pi$, where $\Xi = \{(\xi, 3) | \xi \in [0, \infty)\}$ and $\Pi = \{(\xi, 2) | \xi \in [0, \infty)\}$ such that*

$$\phi(\alpha, 3) = \begin{cases} \left(\frac{\xi}{4}, 2\right) & \xi \in [0, 1) \\ \left(\frac{\xi}{3}, 2\right) & \xi \in [1, \infty) \end{cases},$$

is proximal $\mathcal{Z}$ Górnicki mapping with $\zeta(t, s) = \frac{9}{10}s - t$

Proof. Consider $(\xi, 3), (\omega, 3) \in \Xi$, such that $\xi, \omega \in [0, 1)$. Suppose $\left\|(\frac{\xi}{4}, 2) - (\mu, 3)\right\| = 1$ implies $(\mu, 3) = \left(\frac{\xi}{4}, 3\right)$, similarly for $(\omega, 3)$, $(\nu, 3) = \left(\frac{\omega}{4}, 3\right)$.
Case 1: $\xi, \omega \in [0, 1)$,

$$\begin{aligned} \frac{9}{10}s - t =& \frac{9}{10}\left\|\left(\frac{\xi}{4}, 3\right) - (\xi, 3)\right\| + \left\|\left(\frac{\omega}{4}, 3\right) - (\omega, 3)\right\| + \|(\omega, 3) - (\xi, 3)\| - \left\|\left(\frac{\xi}{4}, 2\right) - \left(\frac{\omega}{4}, 3\right)\right\|, \\ =& \frac{9}{10}\left(\frac{3\xi}{4} + \frac{3\omega}{4}\right) + \frac{13}{20}|\xi - \omega| \geq 0. \end{aligned}$$

Case 2: $\xi, \omega \in [1, \infty)$,

$$\frac{9}{10}s - t = \frac{9}{10}\left(\frac{2\xi}{3} + \frac{2\omega}{3}\right) + \frac{17}{30}|\xi - \omega| \geq 0.$$

Case 3: $\xi \in [0, 1), \omega \in [1, \infty)$. Then $\xi - \omega \leq 0$,

$$\frac{9}{10}s - t = \frac{\xi}{40} + \frac{7\omega}{6} \geq 0.$$

Case 4: $\omega \in [0, 1), \xi \in [1, \infty)$. Then $\xi - \omega \geq 0$,

$$\frac{9}{10}s - t = \frac{\omega}{40} + \frac{7\xi}{6} \geq 0.$$

Iteration of convergence for a proximal $\mathcal{Z}-$ Górnicki mapping

ξ_n	$\xi_0 = (0.8, 3)$	$\xi_0 = (2, 3)$	$\xi_0 = (3.5, 3)$	$\xi_0 = (6, 3)$
ξ_1	(0.8,3)	(2,3)	(3.5,3)	(6,3)
ξ_2	(4.0000e-01,3)	(1,3)	(1.7500e+00,3)	(3,3)
ξ_3	(2.0000e-01,3)	(5.0000e-01,3)	(8.7500e-01,3)	(1.5000e+00,3)
ξ_4	(1.0000e-01,3)	(2.5000e-01,3)	(4.3750e-01,3)	(7.5000e-01,3)
ξ_5	(5.0000e-02,3)	(1.2500e-01,3)	(2.1875e-01,3)	(3.7500e-01,3)
ξ_6	(2.5000e-02,3)	(6.2500e-02,3)	(1.0937e-01,3)	(1.8750e-01,3)
ξ_7	(1.2500e-02,3)	(3.1250e-02,3)	(5.4687e-02,3)	(9.3750e-02,3)
ξ_8	(6.2500e-03,3)	(1.5625e-02,3)	(2.7343e-02,3)	(4.6875e-02,3)
ξ_9	(3.1250e-03,3)	(7.8125e-03,3)	(1.36718e-02,3)	(2.3437e-02,3)
ξ_{10}	(1.5625e-03,3)	(3.9062e-03,3)	(6.8359e-03,3)	(1.1718e-02,3)
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
ξ_{100}	(1.2621e-30,3)	(3.1554e-30,3)	(5.5220e-30,3)	(9.4663e-30,3)

consider $(\xi, 3) \in \Xi$ where $\xi \in [0, 1)$ with $(\mu, 3) = \left(\frac{\xi}{4}, 3\right)$, take $(\gamma, 3) = \left(\frac{\xi}{2}, 3\right)$ suppose $\|(\phi\gamma, 3) - (\rho, 3)\| = 1$ then $\rho = \left(\frac{\xi}{8}, 3\right)$ implies

$$\frac{9}{10}\left\|(\xi, 3) - \left(\frac{\xi}{4}, 3\right)\right\| - \left\|\left(\frac{\xi}{2}, 3\right) - (\xi, 3)\right\| = \frac{7\xi}{40} \geq 0$$
$$\frac{9}{10}\left\|(\xi, 3) - \left(\frac{\xi}{4}, 3\right)\right\| - \left\|\left(\frac{\xi}{2} - 3\right) - \left(\frac{\xi}{8}, 3\right)\right\| = \frac{12\xi}{40} \geq 0,$$

consider $(\xi, 3) \in \Xi$ where $\xi \in [1, \infty)$ with $(\mu, 3) = \left(\frac{\xi}{3}, 3\right)$, take $(\gamma, 3) = \left(\frac{\xi}{2}, 3\right)$, suppose $\|(\phi\gamma, 3) - (\rho, 3)\| = 1$ with $\frac{\xi}{2} < 1$ then $(\rho, 3) = \left(\frac{\xi}{8}, 3\right)$ implies

$$\frac{9}{10}\left\|(\xi, 3) - \left(\frac{\xi}{3}, 3\right)\right\| - \left\|\left(\frac{\xi}{2}, 3\right) - (\xi, 3)\right\| = \frac{3\xi}{30} \geq 0,$$
$$\frac{9}{10}\left\|(\xi, 3) - \left(\frac{\xi}{3}, 3\right)\right\| - \left\|\left(\frac{\xi}{2}, 3\right) - \left(\frac{\xi}{8}, 3\right)\right\| = \frac{9\xi}{40} \geq 0,$$

suppose $\|(\phi\gamma, 3), (\rho, 3)\| = 1$ with $\frac{\xi}{2} \geq 1$ then $(\rho, 3) = \left(\frac{\xi}{6}, 3\right)$ implies

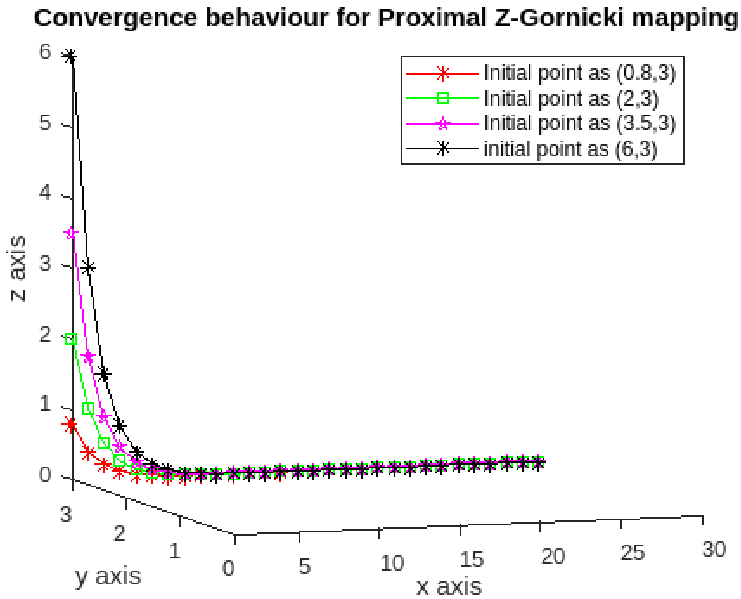
$$\frac{9}{10}\left\|(\xi, 3) - \left(\frac{\xi}{3}, 3\right)\right\| - \left\|\left(\frac{\xi}{2}, 3\right) - (\xi, 3)\right\| = \frac{3\xi}{30} \geq 0,$$
$$\frac{9}{10}\left\|(\xi, 3) - \left(\frac{\xi}{3}, 3\right)\right\| - \left\|\left(\frac{\xi}{2}, 3\right) - \left(\frac{\xi}{6}, 3\right)\right\| = \frac{8\xi}{30} \geq 0,$$

Therefore ϕ is proximal $\mathcal{Z}-$ Górnicki mapping with respect to ζ where $\tau = 1$. □

Corollary 3.1. *Let Ξ be a nonempty closed subset of complete metric spaces (Γ, κ). A mapping $\phi : \Xi \to \Xi$ is said to be proximal $\mathcal{Z}-$ Górnicki mapping with respect to ζ then ϕ is $\mathcal{Z}-$ Górnicki mapping with respect to some ζ.*

Definition 3.2. *Let Ξ and Π be nonempty subsets of complete partial metric space (Γ, h). A mapping $\phi : \Xi \to \Pi$ is said to be proximal partial $\mathcal{Z}-$ Górnicki mapping with respect to ζ in $\mathcal{Z}$ if for every $\xi, \omega, \mu, \nu \in \Xi$ whenever*

$$h(\phi\xi, \mu) = h(\Xi, \Pi), \quad \textit{and}$$
$$h(\phi\omega, \nu) = h(\Xi, \Pi),$$



imply

$$\zeta\left(h(\mu,\nu),h(\xi,\mu)+h(\xi,\omega)+h(\omega,\nu)\right)\geq 0, \tag{BZ1}$$

there exists $\tau\in[0,\infty)$ *such that for given any* $\xi\in\Xi$ *there exists* $\gamma\in\Xi$ *whenever* $h(\phi\gamma,\rho)=h(\Xi,\Pi)$ *implies that*

$$\zeta\left(h(\gamma,\rho),h(\xi,\mu)\right)\geq 0, \quad \text{and} \tag{BZ2}$$

$$\zeta\left(h(\xi,\gamma),\tau\, h(\xi,\mu)\right)\geq 0, \tag{BZ3}$$

where $\rho\in\Xi$.

Theorem 3.2. *Let* Ξ *and* Π *be nonempty closed subsets of complete partial metric space* (Γ,h)*. If a mapping* $\phi:\Xi\to\Pi$ *is proximal partial* $\mathcal{Z}-$ *Górnicki mapping with respect to* ζ*. Then* ϕ *has a best proximity point.*

Proof. Use a procedure similar to Theorem 3.1 and Theorem 2.3. □

Corollary 3.2. *Let* Ξ *be a nonempty closed subset of complete partial metric spaces* (Γ,h)*. Let mapping* $\phi:\Xi\to\Xi$ *be proximal partial* $\mathcal{Z}-$ *Górnicki mapping with respect to* ζ*. Then* ϕ *is* $\mathcal{Z}-$ *Górnicki mapping with respect to* ζ*.*

4 Application to The Static Deflection of a String Stretched Under Unit Tension Between Fixed Endpoints

In this section, we examine the static deflection of a string held under unit tension between fixed endpoints, subjected to a force distribution $f(\xi)$ (in Newtons per unit length).

The static deflection of the string transformed into a boundary value problem is as follows:

$$\omega''(\xi)=f(\xi) \text{ with } \omega(0)=0\,,\ \omega(l)=0. \tag{4.1}$$

Suppose that $f(\xi) = 0$ and l is the length of the stretched string. Consider Green's function for the given differential equation

$$G(\xi, \varpi) = \begin{cases} \dfrac{\varpi(\xi - l)}{l} & 0 \leq \varpi \leq \xi \\ \xi(\varpi - l) & \xi \leq \varpi \leq l. \end{cases}$$

The differential equation leads to the following Fredholm integral equation,

$$\omega(\xi) = \int_0^l G(\xi, \varpi)\omega(\varpi)d\varpi.$$

Theorem 4.1. *Consider the mapping $\phi : \Phi \to \Phi$ such that*

$$\phi\omega(\xi) = \int_0^l G(\xi, \varpi)\omega(\varpi)d\varpi, \tag{4.2}$$

with $\sup \left| \int_0^l G(\xi, \varpi)\omega(\varpi)d\varpi \right| \leq \frac{1}{2}$, where $\Phi = C[0, l]$. Then the Fredholm integral equation (4.2) admits a unique fixed point, which corresponds to a solution of equation (4.1).

Proof. Here ϕ is $\mathcal{Z}$- Gornicki mapping with respect to $\zeta = \frac{1}{2}\varpi - \gamma$. Define metric as

$$\kappa(\xi(\varpi), \omega(\varpi)) = \begin{cases} \sup\limits_{\varpi \in [0,1]} |\xi(\varpi)| + \sup\limits_{\varpi \in [0,1]} |\omega(\varpi)| & \xi(\varpi) \neq \omega(\varpi) \\ 0 & \xi(\varpi) = \omega(\varpi) \ \forall \, \xi \in [0, 1]. \end{cases}$$

Then

$$\begin{aligned} \kappa(\phi\xi(\varpi), \phi\omega(\varpi)) &= \sup_{\varpi \in [0,1]} \left| \int_0^1 G(\xi, \varpi)\xi(\varpi) \right| + \sup_{\varpi \in [0,1]} \left| \int_0^1 G(\xi, \varpi)\omega(\varpi) \right|, \\ &\leq \frac{1}{2} \sup_{\varpi \in [0,1]} |\xi(\varpi)| + \frac{1}{2} \sup_{\varpi \in [0,1]} |\omega(\varpi)|, \\ &= \frac{1}{2}\kappa(\xi(\varpi), \omega(\varpi)), \end{aligned}$$

implies $\zeta(\kappa(\phi\xi(\varpi), \phi\omega(\varpi)), \kappa(\xi(\varpi), \phi\xi(\varpi)) + \kappa(\xi(\varpi), \omega(\varpi)) + \kappa(\omega(\varpi), \phi\omega(\varpi))) \geq 0$.
Consider $m = \sup \left| \int_0^1 G(\xi, \varpi)\xi(\varpi)d\varpi \right|$, $l = \sup\limits_{\varpi \in [0,1]} |\xi(\varpi)|$ and $n = \max\{2, l, m\}$.
Then by taking $\mu(\varpi) = \frac{\xi(\varpi)}{n}$,

$$\frac{1}{2}\kappa(\xi(\varpi), \phi\xi(\varpi)) - \kappa(\mu(\varpi), \phi\mu(\varpi)) = \frac{n-2}{2n} \left(\sup_{\varpi \in [0,1]} \left| \int_0^1 G(\xi, \varpi)\xi(\varpi)d\varpi \right| + \sup_{\varpi \in [0,1]} |\xi(\varpi)| \right) \geq 0,$$

and also

$$\begin{aligned} \kappa(\xi(\varpi), \phi\xi(\varpi)) &= \left(\sup_{\varpi \in [0,1]} \left| \int_0^1 G(\xi, \varpi)\xi(\varpi)dv \right| + \sup_{\varpi \in [0,1]} |\xi(\varpi)| \right), \\ &= m + \sup_{\varpi \in [0,1]} |\xi(\varpi)|. \end{aligned}$$

If $m \geq l$,

$$\begin{aligned}\kappa(\xi(\varpi), \phi\xi(\varpi)) &\geq \frac{l}{n} + \sup_{\varpi\in[0,1]} |\xi(\varpi)|, \\ &= \kappa(\xi(\varpi), \mu(\varpi)).\end{aligned}$$

Otherwise, choose $n \in \mathbb{N}$ such that $m \geq \frac{l}{n}$. Then above equations hold.

Therefore ϕ is a $\mathcal{Z}-$ Górnicki mapping with respect to $\omega(\gamma, \varpi) = \frac{1}{2}\varpi - \gamma$, where $\tau = 2$. $\square$

Conflicts of Interest The authors report no conflicts of interest.

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